Journal of Mathematical Sciences: Advances and Applications
Volume 15, Number 1, 2012, Pages 37-53

RECENT ADVANCES IN THE CONSTRUCTION OF GREEN'S FUNCTIONS FOR REGIONS ON SURFACES OF REVOLUTION

VOLODYMYR BORODIN

Computational Science Program
Middle Tennessee State University
Tennessee
USA
e-mail: vb2m@mtmail.mtsu.edu

Abstract

A technique, which is based on the separation of variables with subsequent either complete or partial summation of the resulting series, is used to obtain a number of computer-friendly representations of Green's functions. Boundary-value problems, stated for the Laplace equation are considered in regions on a spherical surface. An extension of the technique to other surfaces of revolution is discussed.

1. Introduction

The Green's function approach is widely used in nowadays for the solution of applied boundary-value problems for partial differential equations [2, 9, 10]. Green's function could be especially important for developing the boundary-element method numerical techniques [3-5]. In this study, we aim at the extension of the technique, proposed in [8], to a

2010 Mathematics Subject Classification: 35J08, 35J05, 35J25.

Keywords and phrases: Green's function, Laplace equation, boundary-value problems, geographical coordinates, analytical solution.

Received April 23, 2012

number of problems applicable in those areas of engineering and science that deal with thin shells.

We will start with the construction of the Green's function of a boundary-value problem stated for the Laplace equation in a region Ω on a spherical surface of radius a. That is, the Poisson equation

$$\frac{1}{a^2}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial u(\varphi,\theta)}{\partial\varphi}\right)+\frac{1}{a^2\sin\varphi}\frac{\partial^2 u(\varphi,\theta)}{\partial\theta^2}=-f(\varphi,\theta),\qquad \varphi,\theta\in\Omega, \quad (1)$$

written in geographical coordinates is subject to the boundary conditions

$$B_1[u(\varphi,\theta_1)]=0,\quad B_2[u(\varphi,\theta_2)]=0; \qquad (2)$$

and

$$B_3[u(\varphi_1,\theta)]=0,\quad B_4[u(\varphi_2,\theta)]=0, \qquad (3)$$

where B_i, $i=\overline{1,4}$, are the boundary condition operators.

If $G(\varphi,\theta;\psi,\tau)$ represents the Green's function of the homogeneous boundary-value problem corresponding to (1)-(3), then the solution to the problem in (1)-(3) itself can be expressed as the volume integral

$$u(\varphi,\theta)=\int_\Omega G(\varphi,\theta;\psi,\tau)f(\psi,\tau)d_{\psi,\tau}\Omega. \qquad (4)$$

Assume that the boundary-value problem in (1)-(3) allows separation of variables. Implying that B_1 and B_2 represent a combination of Dirichlet and Neumann operators, we expand the solution $u(\varphi,\theta)$ of the original problem and the right-hand side function $f(\varphi,\theta)$ of the governing equation as

$$u(\varphi,\theta)=\sum_{n=1}^{\infty}u_n(\varphi)\sin\nu\theta, \qquad (5)$$

and

$$f(\varphi, \theta) = \sum_{n=1}^{\infty} f_n(\varphi) \sin \nu\theta, \tag{6}$$

where the factor ν is directly proportional to the index of summation n.

Substituting the above trigonometric representations into the original boundary-value problem, we obtain the set

$$\frac{1}{a^2} \frac{d}{d\varphi}\left(\sin \varphi \frac{du_n(\varphi)}{d\varphi}\right) - \frac{\nu^2}{a^2 \sin \varphi} u_n(\varphi) = -f_n(\varphi), \quad n = 1, 2, 3, \dots \tag{7}$$

$$B_3[u_n(\varphi_1)] = 0, \quad B_4[u_n(\varphi_2)] = 0, \tag{8}$$

of boundary-value problems for ordinary differential equations.

Keeping in mind the application of the method of variation of parameters to the boundary-value problems in (7)-(8), we need two linearly independent particular solutions for the homogeneous equation corresponding to (7). Obtaining those, we change the independent variable φ as

$$\omega = \ln\left(\tan\left(\frac{\varphi}{2}\right)\right),$$

which reduces the problem in (7)-(8) to

$$\frac{d^2 u_n(\omega)}{d\omega^2} - \nu^2 u_n(\omega) = 0, \tag{9}$$

$$B_3[u_n(\omega_1)] = 0, \quad B_4[u_n(\omega_2)] = 0. \tag{10}$$

This allows, within the scope of the method of variation of parameters, the general solution for (9) in the form

$$u_n(\omega) = C_1(\omega) e^{\nu\omega} + C_2(\omega) e^{-\nu\omega},$$

or going back to the variable φ, we have the solution $u_n(\omega)$ in form

$$u_n(\varphi) = C_1(\varphi) \tan^{\nu}\left(\frac{\varphi}{2}\right) + C_2(\varphi) \tan^{-\nu}\left(\frac{\varphi}{2}\right). \tag{11}$$

Following the procedure of the method, we arrive at the system of linear algebraic equations

$$\begin{cases} C_1'(\varphi)\tan^{\nu}\left(\frac{\varphi}{2}\right) + C_2'(\varphi)\tan^{-\nu}\left(\frac{\varphi}{2}\right) = 0, \\ C_1'(\varphi)\nu\tan^{\nu}\left(\frac{\varphi}{2}\right) - C_2'(\varphi)\nu\tan^{-\nu}\left(\frac{\varphi}{2}\right) = a^2 f(\varphi)\sin(\varphi), \end{cases}$$

in $C_1'(\varphi)$ and $C_2'(\varphi)$, whose solution can be written in the form

$$C_1'(\varphi) = -\frac{\widetilde{f}_n(\varphi)}{2\nu\tan^{\nu}(\varphi/2)},$$

and

$$C_2'(\varphi) = \frac{\widetilde{f}_n(\varphi)}{2\nu\tan^{-\nu}(\varphi/2)},$$

where $\widetilde{f}_n(\varphi) = a^2 f_n(\varphi)\sin\varphi$.

This gives rise to

$$C_1(\varphi) = -\int_{\varphi_1}^{\varphi}\frac{1}{2\nu\tan^{\nu}(\psi/2)}\widetilde{f}_n(\psi)d\psi + D_1,$$

and

$$C_2(\varphi) = \int_{\varphi_1}^{\varphi}\frac{1}{2\nu\tan^{-\nu}(\psi/2)}\widetilde{f}_n(\psi)d\psi + D_2.$$

Upon substitution of these into (11), we obtain

$$u_n(\varphi) = -\int_{\varphi_1}^{\varphi}\frac{\tan^{\nu}(\varphi/2)}{2\nu\tan^{\nu}(\psi/2)}\widetilde{f}_n(\psi)d\psi + D_1\tan^{\nu}(\varphi/2)$$

$$+\int_{\varphi_1}^{\varphi}\frac{\tan^{\nu}(\psi/2)}{2\nu\tan^{\nu}(\varphi/2)}\widetilde{f}_n(\psi)d\psi + D_2\tan^{-\nu}(\varphi/2),$$

which transforms into

$$u_n(\varphi) = \frac{1}{2\nu}\int_{\varphi_1}^{\varphi}\left(\frac{\tan^{\nu}(\psi/2)}{\tan^{\nu}(\varphi/2)} - \frac{\tan^{\nu}(\varphi/2)}{\tan^{\nu}(\psi/2)}\right)\widetilde{f}_n(\psi)d\psi \tag{12}$$

$$+ D_1 \tan^{\nu}(\varphi / 2) + D_2 \tan^{-\nu}(\varphi / 2).$$

Satisfying the boundary conditions in (8), we will be able to express (12) in a form of a simple integral

$$u_n(\varphi) = \int_{\varphi_1}^{\varphi_2} g_n(\varphi, \psi)\tilde{f}_n(\psi)d\psi, \tag{13}$$

where the kernel $g_n(\varphi, \psi)$ is expressed in two pieces. In considering particular problems in sections that follow, we discuss this last issue in detail.

To proceed further with our approach, we express the coefficients $f_n(\psi)$ of (6) by using the Euler-Fourier formula

$$f_n(\varphi) = \frac{2}{\theta_2 - \theta_1}\int_{\theta_1}^{\theta_2} f(\varphi, \vartheta)\sin \nu\vartheta d\vartheta,$$

and substitute $u_n(\varphi)$ into (5). This yields

$$u(\varphi, \theta) = \int_{\Omega} G(\varphi, \theta; \psi, \tau) f(\psi, \tau) d_{\psi, \tau}\Omega,$$

where $G(\varphi, \theta; \psi, \tau)$ represents the Green's function of the homogeneous boundary-value problem corresponding to (1)-(3) [6] that appears in the form

$$G(\varphi, \theta; \psi, \tau) = \frac{2}{\theta_2 - \theta_1}\sum_{n=1}^{\infty} g_n(\varphi, \psi)\sin \nu\theta \sin \nu\tau. \tag{14}$$

In the following sections, we consider a number of particular problem settings.

2. Dirichlet Problem on a Spherical Triangle

Consider the spherical triangle $\Omega = \{\varphi, \theta \,|\, 0 \le \varphi \le \beta;\ 0 \le \theta \le \gamma\}$, where $0 < \beta < \pi$ and $0 < \gamma < 2\pi$. For Dirichlet boundary conditions, with B_i, $i = \overline{1, 4}$ in (2)-(3) being the identity operators, we have

$$B_1[u(\varphi,\, \theta_1)] \equiv u(\varphi,\, 0) = 0, \quad B_2[u(\varphi,\, \theta_2)] \equiv u(\varphi,\, \gamma) = 0, \tag{15}$$

$$B_3[u(\varphi_1,\, \theta)] \equiv |u(0,\, \theta)| < \infty, \quad B_4[u(\varphi_2,\, \theta)] \equiv u(\beta,\, \theta) = 0. \tag{16}$$

To satisfy the boundary conditions in (15), the parameter ν in the Fourier series expansions of (5)-(6) must be $\nu = n\pi / \gamma$. By satisfying the boundary conditions in (16), we determine the constants D_1 and D_2 in (12). First, the boundedness condition $|u(0,\, \theta)| < \infty,$ provides us with

$$D_2 = 0. \tag{17}$$

The second condition results into

$$\frac{1}{2\nu}\int_0^{\beta}\left(\frac{\tan^{\nu}(\psi/2)}{\tan^{\nu}(\beta/2)} - \frac{\tan^{\nu}(\beta/2)}{\tan^{\nu}(\psi/2)}\right)\tilde{f}_n(\psi)d\psi + D_1 \tan^{\nu}(\beta/2) = 0.$$

Introducing the shorthand notations

$$\Phi(\xi) = \tan^{\pi/\gamma}(\xi/2),$$

and

$$B = \tan^{\pi/\gamma}(\beta/2),$$

the above expression can be solved for D_1

$$D_1 = \frac{1}{2\nu}\int_0^{\beta}\frac{1}{B^n}\left(\frac{B^n}{\Phi^n(\psi)} - \frac{\Phi^n(\psi)}{B^n}\right)\tilde{f}_n(\psi)d\psi. \tag{18}$$

Finally, substituting expressions from (17) and (18) into (12), one obtains

$$u_n(\varphi) = \frac{1}{2\nu}\int_0^{\varphi}\frac{\Phi^n(\psi)}{B^n}\left(\frac{B^n}{\Phi^n(\varphi)} - \frac{\Phi^n(\varphi)}{B^n}\right)\tilde{f}_n(\psi)d\psi$$

$$+ \frac{1}{2\nu}\int_{\varphi}^{\beta}\frac{\Phi^n(\varphi)}{B^n}\left(\frac{B^n}{\Phi^n(\psi)} - \frac{\Phi^n(\psi)}{B^n}\right)\tilde{f}_n(\psi)d\psi,$$

which could be interpreted as (13), where

$$g_n(\varphi, \psi) = \frac{1}{2\nu}\begin{cases} \dfrac{\Phi^n(\psi)}{B^n}\left(\dfrac{\Phi^n(\varphi)}{B^n} - \dfrac{B^n}{\Phi^n(\varphi)}\right), & \text{if } 0 \leq \psi \leq \varphi, \\ \dfrac{\Phi^n(\varphi)}{B^n}\left(\dfrac{\Phi^n(\psi)}{B^n} - \dfrac{B^n}{\Phi^n(\psi)}\right), & \text{if } \varphi \leq \psi \leq \beta. \end{cases}$$

Now, we make use of formula (14) to find the Green's function for boundary-value problem corresponding to that in (1), (15), and (16). Breaking product of sines into the difference of cosines and using the standard summation formula [1, 7]

$$\sum_{n=1}^{\infty} \frac{p^n}{n} \cos(n\alpha) = -\ln\left(\sqrt{1 - 2p\cos\alpha + p^2}\right),$$

the Green's function in (14) could be expressed as

$$G(\varphi, \theta; \psi, \tau) = -\frac{1}{4\pi}\ln\left(\frac{B^4 - 2B^2\Phi(\varphi)\Phi(\psi)\cos(\kappa(\theta, \tau)) + \Phi^2(\varphi)\Phi^2(\psi)}{B^4 - 2B^2\Phi(\varphi)\Phi(\psi)\cos(\eta(\theta, \tau)) + \Phi^2(\varphi)\Phi^2(\psi)}\right)$$

$$+ \frac{1}{4\pi}\ln\left(\frac{\Phi^2(\varphi) - 2\Phi(\varphi)\Phi(\psi)\cos(\kappa(\theta, \tau)) + \Phi^2(\psi)}{\Phi^2(\varphi) - 2\Phi(\varphi)\Phi(\psi)\cos(\eta(\theta, \tau)) + \Phi^2(\psi)}\right), \quad (19)$$

where we again use, for shorthand,

$$\Phi(\xi) = \tan^{\pi/\gamma}(\xi / 2), \quad B = \tan^{\pi/\gamma}(\beta / 2);$$

and

$$\kappa(\theta, \tau) = \frac{\pi}{\gamma}(\theta + \tau), \quad \eta(\theta, \tau) = \frac{\pi}{\gamma}(\theta - \tau).$$

The Green's function in (19) possesses the logarithmic singularity when $\varphi \to \psi$ and $\theta \to \tau$ and represents the solution of the boundary-value problem in (1), (15), and (16), if the right-hand side function $f(\varphi, \theta)$ is understood as the Dirack delta-function $f(\varphi, \theta) = \delta(\varphi - \psi, \theta - \tau)$.

3. Other Boundary-Value Problems Posed on Sphere

Varying the boundary condition operators in (2)-(3) and the domain for the variables φ and θ, and using the technique described in the previous section, we can obtain a number of Green's functions for the Laplace equation posed on sphere.

Before proceeding further, for convenience, we make some simplifying assumptions. First, a 4-letter scheme will be used to specify the boundary conditions in a short way. Each letter corresponds to a boundary condition operator in (2)-(3). "D" relates to the Dirichlet condition, while "N" means the Neumann condition. "S" means the boundary condition at a singular point. For example, the case of boundary conditions, considered in the previous section, could be specified as DDSD. In addition, we introduce two logarithmic functions

$$H_D(x,\ \alpha,\ \beta) = -\frac{1}{4\pi}\ln\left(\frac{1-2x\cos\alpha+x^2}{1-2x\cos\beta+x^2}\right),$$

and

$$H_N(x,\ \alpha,\ \beta) = +\frac{1}{4\pi}\ln\left(\frac{1+2x\cos\alpha+x^2}{1-2x\cos\alpha+x^2}\right)-\frac{1}{4\pi}\ln\left(\frac{1+2x\cos\beta+x^2}{1-2x\cos\beta+x^2}\right).$$

Using such a specification scheme, we present, in Tables 1 and 2, a list of Green's functions for a number of well-posed boundary conditions. For cases, where the series in (14) cannot be totally summed up, we will split the logarithmic singularity, and leave the regular components R_N^i, expressed as uniformly convergent series. These are presented for the corresponding problems later.

Table 1. Green's functions for boundary-value problems posed on sphere

#	Boundary conditions	Ω	Green's function
1	DDSD	$\{\varphi, \theta \mid 0 \le \varphi \le \beta;\ 0 \le \theta \le \gamma\}$	$H_D\left(\frac{\Phi(\varphi)\Phi(\psi)}{B^2}, \kappa, \eta\right) - H_D\left(\frac{\Phi(\varphi)}{\Phi(\psi)}, \kappa, \eta\right)$
2	DDSN	$\{\varphi, \theta \mid 0 \le \varphi \le \beta;\ 0 \le \theta \le \gamma\}$	$-H_D\left(\frac{\Phi(\varphi)\Phi(\psi)}{B^2}, \kappa, \eta\right) - H_D\left(\frac{\Phi(\varphi)}{\Phi(\psi)}, \kappa, \eta\right)$
3	DNSD	$\{\varphi, \theta \mid 0 \le \varphi \le \beta;\ 0 \le \theta \le \gamma\}$	$H_N\left(\sqrt{\frac{\Phi(\varphi)\Phi(\psi)}{B^2}}, \frac{\kappa}{2}, \frac{\eta}{2}\right) - H_N\left(\sqrt{\frac{\Phi(\varphi)}{\Phi(\psi)}}, \frac{\kappa}{2}, \frac{\eta}{2}\right)$
4	DNSN	$\{\varphi, \theta \mid 0 \le \varphi \le \beta;\ 0 \le \theta \le \gamma\}$	$-H_N\left(\sqrt{\frac{\Phi(\varphi)\Phi(\psi)}{B^2}}, \frac{\kappa}{2}, \frac{\eta}{2}\right) - H_N\left(\sqrt{\frac{\Phi(\varphi)}{\Phi(\psi)}}, \frac{\kappa}{2}, \frac{\eta}{2}\right)$
5	DDSS	$\{\varphi, \theta \mid 0 \le \varphi \le \pi;\ 0 \le \theta \le \gamma\}$	$-H_D\left(\frac{\Phi(\varphi)}{\Phi(\psi)}, \kappa, \eta\right)$
6	DNSS	$\{\varphi, \theta \mid 0 \le \varphi \le \pi;\ 0 \le \theta \le \gamma\}$	$-H_N\left(\frac{\Phi(\varphi)}{\Phi(\psi)}, \kappa, \eta\right)$
7	DDDD	$\{\varphi, \theta \mid \alpha \le \varphi \le \beta;\ 0 \le \theta \le \gamma\}$	$H_D\left(\frac{\Phi(\varphi)\Phi(\psi)}{B^2}, \kappa, \eta\right) - H_D\left(\frac{\Phi(\varphi)A^2}{\Phi(\psi)B^2}, \kappa, \eta\right)$ $-H_D\left(\frac{\Phi(\varphi)}{\Phi(\psi)}, \kappa, \eta\right) + H_D\left(\frac{A^2}{\Phi(\varphi)\Phi(\psi)}, \kappa, \eta\right) + R_N^7$
8	DDDN	$\{\varphi, \theta \mid \alpha \le \varphi \le \beta;\ 0 \le \theta \le \gamma\}$	$-H_D\left(\frac{\Phi(\varphi)\Phi(\psi)}{B^2}, \kappa, \eta\right) + H_D\left(\frac{\Phi(\varphi)A^2}{\Phi(\psi)B^2}, \kappa, \eta\right)$ $-H_D\left(\frac{\Phi(\varphi)}{\Phi(\psi)}, \kappa, \eta\right) + H_D\left(\frac{A^2}{\Phi(\varphi)\Phi(\psi)}, \kappa, \eta\right) + R_N^8$
9	DDND	$\{\varphi, \theta \mid \alpha \le \varphi \le \beta;\ 0 \le \theta \le \gamma\}$	$H_D\left(\frac{\Phi(\varphi)\Phi(\psi)}{B^2}, \kappa, \eta\right) + H_D\left(\frac{\Phi(\varphi)A^2}{\Phi(\psi)B^2}, \kappa, \eta\right)$ $-H_D\left(\frac{\Phi(\varphi)}{\Phi(\psi)}, \kappa, \eta\right) - H_D\left(\frac{A^2}{\Phi(\varphi)\Phi(\psi)}, \kappa, \eta\right) + R_N^9$

Table 2. Green's functions for boundary-value problems posed on sphere (continuation)

#	Boundary conditions	Ω	Green's function
10	DDNN	$\{\varphi, \theta \mid \alpha \le \varphi \le \beta;\ 0 \le \theta \le \gamma\}$	$-H_D\left(\frac{\Phi(\varphi)\Phi(\psi)}{B^2}, \kappa, \eta\right) - H_D\left(\frac{\Phi(\varphi)A^2}{\Phi(\psi)B^2}, \kappa, \eta\right)$ $-H_D\left(\frac{\Phi(\varphi)}{\Phi(\psi)}, \kappa, \eta\right) - H_D\left(\frac{A^2}{\Phi(\varphi)\Phi(\psi)}, \kappa, \eta\right) + R_N^{10}$
11	DNDD	$\{\varphi, \theta \mid \alpha \le \varphi \le \beta;\ 0 \le \theta \le \gamma\}$	$H_N\left(\sqrt{\frac{\Phi(\varphi)\Phi(\psi)}{B^2}}, \frac{\kappa}{2}, \frac{\eta}{2}\right) - H_N\left(\sqrt{\frac{\Phi(\varphi)A^2}{\Phi(\psi)B^2}}, \frac{\kappa}{2}, \frac{\eta}{2}\right)$ $-H_N\left(\sqrt{\frac{\Phi(\varphi)}{\Phi(\psi)}}, \frac{\kappa}{2}, \frac{\eta}{2}\right) + H_N\left(\sqrt{\frac{A^2}{\Phi(\varphi)\Phi(\psi)}}, \frac{\kappa}{2}, \frac{\eta}{2}\right) + R_N^{11}$
12	DNDN	$\{\varphi, \theta \mid \alpha \le \varphi \le \beta;\ 0 \le \theta \le \gamma\}$	$-H_N\left(\sqrt{\frac{\Phi(\varphi)\Phi(\psi)}{B^2}}, \frac{\kappa}{2}, \frac{\eta}{2}\right) + H_N\left(\sqrt{\frac{\Phi(\varphi)A^2}{\Phi(\psi)B^2}}, \frac{\kappa}{2}, \frac{\eta}{2}\right)$ $-H_N\left(\sqrt{\frac{\Phi(\varphi)}{\Phi(\psi)}}, \frac{\kappa}{2}, \frac{\eta}{2}\right) + H_N\left(\sqrt{\frac{A^2}{\Phi(\varphi)\Phi(\psi)}}, \frac{\kappa}{2}, \frac{\eta}{2}\right) + R_N^{12}$
13	DNND	$\{\varphi, \theta \mid \alpha \le \varphi \le \beta;\ 0 \le \theta \le \gamma\}$	$H_N\left(\sqrt{\frac{\Phi(\varphi)\Phi(\psi)}{B^2}}, \frac{\kappa}{2}, \frac{\eta}{2}\right) + H_N\left(\sqrt{\frac{\Phi(\varphi)A^2}{\Phi(\psi)B^2}}, \frac{\kappa}{2}, \frac{\eta}{2}\right)$ $-H_N\left(\sqrt{\frac{\Phi(\varphi)}{\Phi(\psi)}}, \frac{\kappa}{2}, \frac{\eta}{2}\right) - H_N\left(\sqrt{\frac{A^2}{\Phi(\varphi)\Phi(\psi)}}, \frac{\kappa}{2}, \frac{\eta}{2}\right) + R_N^{13}$
14	DNNN	$\{\varphi, \theta \mid \alpha \le \varphi \le \beta;\ 0 \le \theta \le \gamma\}$	$-H_N\left(\sqrt{\frac{\Phi(\varphi)\Phi(\psi)}{B^2}}, \frac{\kappa}{2}, \frac{\eta}{2}\right) - H_N\left(\sqrt{\frac{\Phi(\varphi)A^2}{\Phi(\psi)B^2}}, \frac{\kappa}{2}, \frac{\eta}{2}\right)$ $-H_N\left(\sqrt{\frac{\Phi(\varphi)}{\Phi(\psi)}}, \frac{\kappa}{2}, \frac{\eta}{2}\right) - H_N\left(\sqrt{\frac{A^2}{\Phi(\varphi)\Phi(\psi)}}, \frac{\kappa}{2}, \frac{\eta}{2}\right) + R_N^{14}$

The regular terms R_N^i, $i = \overline{7, 14}$ for Green's functions, which cannot be totally summed up are shown below:

$$R_N^7 = \frac{1}{2\pi}\sum_{n=1}^{N}\frac{A^{2n}(\Phi^{2n}(\varphi) - B^{2n})(\Phi^{2n}(\psi) - A^{2n})}{B^{2n}\Phi^n(\varphi)\Phi^n(\psi)(B^{2n} - A^{2n})}(\cos n\eta(\theta, \tau) - \cos n\kappa(\theta, \tau)),$$

$$R_N^8 = \frac{1}{2\pi}\sum_{n=1}^{N}\frac{A^{2n}(\Phi^{2n}(\varphi) + B^{2n})(\Phi^{2n}(\psi) - A^{2n})}{B^{2n}\Phi^n(\varphi)\Phi^n(\psi)(B^{2n} + A^{2n})}(\cos n\eta(\theta, \tau) - \cos n\kappa(\theta, \tau)),$$

$$R_N^9 = \frac{1}{2\pi}\sum_{n=1}^{N}\frac{A^{2n}(B^{2n} - \Phi^{2n}(\varphi))(\Phi^{2n}(\psi) + A^{2n})}{B^{2n}\Phi^n(\varphi)\Phi^n(\psi)(B^{2n} + A^{2n})}(\cos n\eta(\theta, \tau) - \cos n\kappa(\theta, \tau)),$$

$$R_N^{10} = \frac{1}{2\pi}\sum_{n=1}^{N}\frac{A^{2n}(\Phi^{2n}(\varphi) + B^{2n})(\Phi^{2n}(\psi) + A^{2n})}{B^{2n}\Phi^n(\varphi)\Phi^n(\psi)(B^{2n} - A^{2n})}(\cos n\eta(\theta, \tau) - \cos n\kappa(\theta, \tau)),$$

$$R_N^{11} = \frac{1}{2\pi}\sum_{n=1}^{N}\frac{A^{n}(\Phi^{n}(\varphi) - B^{n})(\Phi^{n}(\psi) - A^{n})}{B^{n}\sqrt{\Phi^n(\varphi)\Phi^n(\psi)}(B^{n} - A^{n})}\left(\cos\frac{n}{2}\eta(\theta, \tau) - \cos\frac{n}{2}\kappa(\theta, \tau)\right),$$

$$R_N^{12} = \frac{1}{2\pi}\sum_{n=1}^{N}\frac{A^{n}(\Phi^{n}(\varphi) + B^{n})(\Phi^{n}(\psi) - A^{n})}{B^{n}\sqrt{\Phi^n(\varphi)\Phi^n(\psi)}(B^{n} + A^{n})}\left(\cos\frac{n}{2}\eta(\theta, \tau) - \cos\frac{n}{2}\kappa(\theta, \tau)\right),$$

$$R_N^{13} = \frac{1}{2\pi}\sum_{n=1}^{N}\frac{A^{n}(B^{n} - \Phi^{n}(\varphi))(\Phi^{n}(\psi) + A^{n})}{B^{n}\sqrt{\Phi^n(\varphi)\Phi^n(\psi)}(B^{n} + A^{n})}\left(\cos\frac{n}{2}\eta(\theta, \tau) - \cos\frac{n}{2}\kappa(\theta, \tau)\right),$$

and

$$R_N^{14} = \frac{1}{2\pi}\sum_{n=1}^{N}\frac{A^{n}(\Phi^{n}(\varphi) + B^{n})(\Phi^{n}(\psi) + A^{n})}{B^{n}\sqrt{\Phi^n(\varphi)\Phi^n(\psi)}(B^{n} - A^{n})}\left(\cos\frac{n}{2}\eta(\theta, \tau) - \cos\frac{n}{2}\kappa(\theta, \tau)\right).$$

4. 2π-Periodical Problems

Another type of boundary conditions for the problem in (1)-(3) to be considered. It relates to the 2π-periodicity for the coordinate θ, when in (2) $\theta_2 = \theta_1 + 2\pi$. This implies that the boundary condition operators in (2) are written as

$$B_1[u(\varphi,\ \theta_1)] \equiv u(\varphi,\ \theta_1) - u(\varphi,\ \theta_2) = 0,$$

and

$$B_2[u(\varphi,\ \theta_1)] \equiv \frac{\partial u(\varphi,\ \theta_1)}{\partial\theta} - \frac{\partial u(\varphi,\ \theta_2)}{\partial\theta} = 0.$$

In this case, the full Fourier series expansion, for $u(\varphi,\ \theta)$ and $f(\varphi,\ \theta)$ must be used

$$u(\varphi,\ \theta) = \frac{1}{2}u_0(\varphi) + \sum_{n=1}^{\infty} u_n^{(c)}(\varphi)\cos n\theta + \sum_{n=1}^{\infty} u_n^{(s)}(\varphi)\sin n\theta,$$

and

$$f(\varphi,\ \theta) = \frac{1}{2}f_0(\varphi) + \sum_{n=1}^{\infty} f_n^{(c)}(\varphi)\cos n\theta + \sum_{n=1}^{\infty} f_n^{(s)}(\varphi)\sin n\theta.$$

Following the technique described earlier, we express the Green's function in the form

$$G(\varphi,\ \theta;\ \psi,\ \tau) = \frac{1}{2}g_0(\varphi,\ \psi) + \sum_{n=1}^{\infty} g_n^{(c)}(\varphi,\ \psi)\cos n\theta\cos n\tau$$

$$+ \sum_{n=1}^{\infty} g_n^{(s)}(\varphi,\ \psi)\sin n\theta\sin n\tau. \tag{20}$$

The cases $n = 0$ and $n > 0$ must be considered individually. The derivation of the Fourier coefficients is indifferent to the type of the series. That is

$$g_n^{(c)}(\varphi,\ \psi) = g_n^{(s)}(\varphi,\ \psi) = g_n(\varphi,\ \psi), \quad n = 1,\ 2,\ 3,\ \ldots. \tag{21}$$

To derive $g_0(\varphi,\ \psi)$, we find the general solution $u_0(\omega)$ of the equation in (9) for $n = \nu = 0$, which is

$$u_0(\omega) = C_1\omega + C_2,$$

or

$$u_0(\varphi) = C_1 \ln\left(\tan\left(\frac{\varphi}{2}\right)\right) + C_2.$$

Using the method of variation of parameter, one can find the general solution of the corresponding non-homogeneous differential equation with right-hand side function $f_0(\varphi)$ as

$$u_0(\varphi) = -\int_{\varphi_1}^{\varphi} \ln \frac{\Phi_0(\varphi)}{\Phi_0(\psi)} a^2 \sin\psi f_0(\psi)d\psi + D_1 \ln \Phi_0(\varphi) + D_2, \tag{22}$$

where

$$\Phi_0(\xi) = \tan\left(\frac{\xi}{2}\right).$$

Satisfying then the boundary conditions, the above solution reads as

$$u_0(\varphi) = \int_{\varphi_1}^{\varphi_2} g_0(\varphi, \psi) f_0(\psi) d\psi.$$

And the Green's function, that we are looking for, could be written down as

$$G(\varphi, \theta; \psi, \tau) = \frac{1}{2} g_0(\varphi, \psi) + \sum_{n=1}^{\infty} g_n(\varphi, \psi) \cos n(\theta - \tau). \tag{23}$$

We skip detail of a tedious derivation procedure for particular boundary-value problems, and just show the final expressions for their Green's functions.

For the spherical cap: $\Omega = \{\varphi, \theta \mid 0 \le \varphi \le \beta;\ 0 \le \theta \le 2\pi\}$, the boundary conditions are imposed as

$$B_3[u(0, \theta)] \equiv |u(0, \theta)| < \infty,$$

and

$$B_4[u(\beta, \theta)] \equiv u(\beta, \theta) = 0.$$

The ultimate expression for Green's function is found as

$$G(\varphi,\ \theta;\ \psi,\ \tau) = \frac{1}{4\pi}\ln\left(\frac{B_0^2(\Phi_0^2(\varphi) - 2\Phi_0(\varphi)\Phi_0(\psi)\cos(\theta-\tau) + \Phi_0^2(\psi))}{B_0^4 - 2B_0^2\Phi_0(\varphi)\Phi_0(\psi)\cos(\theta-\tau) + \Phi_0^2(\varphi)\Phi_0^2(\psi)}\right),$$

where

$$B_0 = \tan\left(\frac{\beta}{2}\right).$$

For the spherical belt: $\Omega = \{\varphi,\ \theta \,|\, \alpha \le \varphi \le \beta;\ 0 \le \theta \le 2\pi\}$, with the Dirichlet conditions

$$B_3[u(\alpha,\ \theta)] \equiv u(\alpha,\ \theta) = 0,$$

and

$$B_4[u(\beta,\ \theta)] \equiv u(\beta,\ \theta) = 0,$$

the Green's function appears as

$$G(\varphi,\ \theta;\ \psi,\ \tau) = \frac{1}{2}g_0(\varphi,\ \psi) + H_{2\pi}\left(\frac{\Phi_0(\varphi)\Phi_0(\psi)}{B_0^2},\ \theta-\tau\right) - H_{2\pi}\left(\frac{\Phi_0(\varphi)A_0^2}{\Phi_0(\psi)B_0^2},\ \theta-\tau\right)$$

$$- H_{2\pi}\left(\frac{\Phi_0(\varphi)}{\Phi_0(\psi)},\ \theta-\tau\right) + H_{2\pi}\left(\frac{A_0^2}{\Phi_0(\varphi)\Phi_0(\psi)},\ \theta-\tau\right) + R_N,$$

where

$$H_{2\pi}(x,\ \alpha) = \frac{1}{4\pi}\ln(1 - 2x\cos\alpha + x^2),$$

$$g_0(\varphi,\ \psi) = \frac{1}{\ln\frac{A_0}{B_0}}\begin{cases} \ln\frac{B_0}{\Phi_0(\varphi)}\ln\frac{\Phi_0(\psi)}{A_0}, & \text{if } \alpha \le \varphi \le \psi, \\ \ln\frac{B_0}{\Phi_0(\psi)}\ln\frac{\Phi_0(\varphi)}{A_0}, & \text{if } \varphi \le \psi \le \beta, \end{cases}$$

$$R_N = \frac{1}{2\pi}\sum_{n=1}^{N}\frac{A_0^{2n}(\Phi_0^{2n}(\varphi) - B_0^{2n})(\Phi_0^{2n}(\psi) - A_0^{2n})}{B_0^{2n}\Phi_0^n(\varphi)\Phi_0^n(\psi)(B_0^{2n} - A_0^{2n})}\cos n(\theta-\tau),$$

and

$$A_0 = \tan\left(\frac{\alpha}{2}\right).$$

For the spherical belt with the Dirichlet-Neumann conditions imposed as

$$B_3[u(\alpha, \theta)] \equiv u(\alpha, \theta) = 0,$$

and

$$B_4[u(\beta, \theta)] \equiv \frac{\partial u(\beta, \theta)}{\partial \varphi} = 0,$$

we arrive at

$$G(\varphi, \theta; \psi, \tau) = \frac{1}{2} g_0(\varphi, \psi) - H_{2\pi}\left(\frac{\Phi_0(\varphi)\Phi_0(\psi)}{B_0^2}, \theta - \tau\right) + H_{2\pi}\left(\frac{\Phi_0(\varphi)A_0^2}{\Phi_0(\psi)B_0^2}, \theta - \tau\right)$$

$$- H_{2\pi}\left(\frac{\Phi_0(\varphi)}{\Phi_0(\psi)}, \theta - \tau\right) + H_{2\pi}\left(\frac{A_0^2}{\Phi_0(\varphi)\Phi_0(\psi)}, \theta - \tau\right) + R_N,$$

where

$$g_0(\varphi, \psi) = \begin{cases} \ln \frac{\Phi_0(\varphi)}{\Phi_0(\psi)} + \frac{1}{B_0} \ln \frac{A_0}{\Phi_0(\varphi)}, & \text{if } \alpha \le \varphi \le \psi, \\ \frac{1}{B_0} \ln \frac{A_0}{\Phi_0(\varphi)}, & \text{if } \varphi \le \psi \le \beta, \end{cases}$$

and

$$R_N = \frac{1}{2\pi} \sum_{n=1}^{N} \frac{A_0^{2n}(\Phi_0^{2n}(\varphi) + B_0^{2n})(\Phi_0^{2n}(\psi) - A_0^{2n})}{B_0^{2n}\Phi_0^n(\varphi)\Phi_0^n(\psi)(B_0^{2n} + A_0^{2n})} \cos n(\theta - \tau).$$

For the spherical belt with the Neumann-Dirichlet conditions imposed as

$$B_3[u(\alpha, \theta)] = \frac{\partial u(\alpha, \theta)}{\partial \varphi} = 0,$$

and

$$B_4[u(\beta, \theta)] = u(\beta, \theta) = 0,$$

the Green's function reads as

$$G(\varphi,\ \theta;\ \psi,\ \tau) = \frac{1}{2}\, g_0(\varphi,\ \psi) + H_{2\pi}\left(\frac{\Phi_0(\varphi)\Phi_0(\psi)}{B_0^2},\ \theta - \tau\right) + H_{2\pi}\left(\frac{\Phi_0(\varphi)A_0^2}{\Phi_0(\psi)B_0^2},\ \theta - \tau\right)$$

$$- H_{2\pi}\left(\frac{\Phi_0(\varphi)}{\Phi_0(\psi)},\ \theta - \tau\right) - H_{2\pi}\left(\frac{A_0^2}{\Phi_0(\varphi)\Phi_0(\psi)},\ \theta - \tau\right) + R_N,$$

where

$$g_0(\varphi,\ \psi) = \begin{cases} \ln \frac{\Phi_0(\varphi)}{B_0}, & \text{if } \alpha \le \varphi \le \psi, \\ \ln \frac{\Phi_0(\psi)}{B_0}, & \text{if } \varphi \le \psi \le \beta, \end{cases}$$

and

$$R_N = \frac{1}{2\pi} \sum_{n=1}^{N} \frac{A_0^{2n}(B_0^{2n} - \Phi_0^{2n}(\varphi))(\Phi_0^{2n}(\psi) + A_0^{2n})}{B_0^{2n}\Phi_0^{n}(\varphi)\Phi_0^{n}(\psi)(B_0^{2n} + A_0^{2n})} \cos n(\theta - \tau).$$

5. Concluding Remarks

Note that similarly to the developments used in this presentation, computer-friendly forms of Green's functions can be obtained for potential problems set up on other surfaces of revolution. In particular, conical, cylindrical, and toroidal surfaces can be considered.

References

[1] M. Abramovitz and I. Ttegun, Handbook of Mathematical Functions, 10th Edition, National Bureau of Standards, Washington D.C., 1972.

[2] G. Barton, Elements of Green's Function and Propagation, Clarendon Press, Oxford, 1989.

[3] P. K. Benerjee and R. Butterfield, Boundary Element Method in Engineering Science, McGraw-Hill, London, 1981.

[4] J. R. Berger, Boundary element analysis of anisotropic materials with special Green's functions, Engineering Analysis with Boundary Elements 14(2) (1994), 123-131.

[5] C. A. Brebbia, The Boundary Element Method for Engineers, Pentech Press / Halstead Press, London-New York, 1978.

[6] R. Courant and D. Hilbert, Methods of Mathematical Physics, Interscience, New York, 1953.

[7] I. S. Gradstein and I. M. Ryzhik, Tables of Integrals, Series, and Products, Academic Press, New York, 1980.

[8] Yu. A. Melnikov, Influence Functions and Matrices, Marcel Dekker, New York, 1999.

[9] V. D. Sherement, Handbook of Green's Functions and Matrices, WIT Press, Southampton-Boston, 2002.

[10] I. Stackgold, Green's Functions and Boundary-Value Problems, John Wiley, New York, 1980.

■